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A Theory of 1/f Noise

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Let $u(\theta)$ be an absolutely integrable function and define the random process

$$n(t) = \sum_{i=-\infty}^{\infty} u(s_i(t - t_i))$$

where the t_i are Poisson arrivals and the s_i are identically distributed nonnegative random variables. Under routine independence assumptions, one may then calculate a formula for the spectrum of $n(t)$, $S_n(\omega)$, in terms of the probability density of s, $p_s(\alpha)$. If any probability density $p_s(\alpha)$ having the property $p_s(\alpha) \sim 1$ for small α is substituted into this formula, the calculated $S_n(\omega)$ is such that $S_n(\omega) \sim 1/\omega$ for small ω. However, this is not a spectrum of a well-defined random process; here, it is termed a *limit spectrum*. If a probability density having the property $p_s(\alpha) \sim \alpha^\delta$ for small α, where $\delta > 0$, is substituted into the formula instead, a spectrum is calculated which is indeed the spectrum of a well-defined random process. Also, if the latter p_s is suitably close to the former p_s, then the spectrum in the second case approximates, to an arbitrary degree of accuracy, the limit spectrum. It is shown how one may thereby have $1/f$ noise with low-frequency turnover, and also strict $1/f^{1-\delta}$ noise (the latter spectrum being integrable for $\delta > 0$). Suitable examples are given. Actually, $u(\theta)$ may be itself a random process, and the theory is developed on this basis.

KEY WORDS: Flicker effect; 1/f noise; Poisson process.

1. INTRODUCTION

The phenomenon of "$1/f$ noise" is a most ubiquitous one. The first observations of noise processes which had power spectra essentially of the form $1/f$ for frequencies as low as practicably measurable appear to be those of

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Schottky,[1] whose studies were concerned with vacuum tubes; he called it "flicker effect" ("Funkeleffekt"). It has been most widely studied in connection with semiconductors,[2-4] but has also been observed in the frequency fluctuations in quartz crystal oscillators and in seasonal temperature fluctuations,[5] in photographic films,[6] in thin metallic films,[7] and in nerve membranes.[8] The standard mathematical theory has been that of van der Ziel.[9] Mandelbrot[10] has studied some mathematical aspects of $1/f$-like spectra. Offner[11] has studied the problem via computer simulation.

A minor aspect of the problem is that a strict $1/f$ spectrum for small f is not integrable; infinite energy or so-called "infrared catastrophe" is implied. A more important part of the problem has been the apparent difficulty of producing a mathematical model which is not in some respect quite arbitrary: either a relatively strange waveform [e.g., $1/\sqrt{t}$ or $K_0(at)$] or a very special probability density must be postulated to make some theories work.

In this paper, simple mathematical conditions are exhibited which imply a $1/f$ spectrum, and a physical example is given in which no arbitrary assumptions are made. However, no attempt is made here to fit the theory to the various contexts in which $1/f$ noise is observed.

As a preliminary on notation, an expression of the form

$$f(\alpha) \sim g(\alpha) \qquad \text{small} \quad \alpha \tag{1}$$

for continuous functions f, g means

$$\lim_{\alpha \to 0} [f(\alpha)/g(\alpha)] = c_{fg} \tag{2}$$

where c_{fg} is a finite nonzero constant. The expression

$$f(\alpha) \approx g(\alpha) \qquad \text{small} \quad \alpha$$

means that (1) holds in the sense of (2) with $c_{fg} = 1$.

Processes composed of superposed pulses of the form $u(s_i(t - t_i))$ are studied. The $\{t_i\}$ are Poisson arrivals and s_i is a random variable; hence

$$s_i(t - t_i) \tag{3}$$

means the product of s_i and $t - t_i$. Another way of proceeding would be to use $(t - t_i)/T_i$ in place of (3). T_i has the dimension of time and may be thought of as a time constant, and $s_i = 1/T_i$. Theories related to the present one have done the equivalent of carry T_i along in the analysis, but it is believed, for reasons made clear here, that it is more convenient to focus attention on s_i.

2. A POISSON PROCESS

Let $w(t)$ be a random process, and define

$$\bar{w}(t) = E\{w(t)\}, \qquad w_a(t) = |w(t)|, \qquad \bar{w}_a(t) = E\{w_a(t)\} \tag{4}$$

and let

$$\mathscr{L}_w(\tau) = \int_{-\infty}^{\infty} E\{w(t)\, w(t+\tau)\}\, dt \tag{5}$$

Let $\{t_i\}$ be Poisson arrivals with average frequency λ. Let $\{w_i(t)\}$ be a sequence of random processes, all having the distribution of $w(t)$, and all being mutually independent of each other and of the arrival times. Define the random process

$$n(t) = \sum_{i=-\infty}^{\infty} w_i(t - t_i) \tag{6}$$

It is shown in Appendix A that under the assumption that

$$\int_{-\infty}^{\infty} \bar{w}_a(t)\, dt < \infty$$

the process $n(t)$ is well-defined and stationary, with mean

$$\bar{n} = \lambda \int_{-\infty}^{\infty} \bar{w}(t)\, dt$$

and covariance

$$\mathscr{L}_n(\tau) = \lambda \mathscr{L}_w(\tau)$$

where $\bar{w}$ is defined in Eqs. (4) and $\mathscr{L}_w$ in Eq. (5).

Now, let $u(\theta)$ be a random process, θ dimensionless, define $u_a(\theta) = |u(\theta)|$ and assume

$$\int_{-\infty}^{\infty} \bar{u}_a(\theta)\, d\theta < \infty \tag{7}$$

Hence by the preceding results,

$$\mathscr{L}_u(\mu) = \int_{-\infty}^{\infty} E\{u(\theta)\, u(\theta + \mu)\}\, d\theta$$

has the form of a covariance function; for example, it is continuous. It is further assumed that $\mathscr{L}_u(\mu)$ is of bounded variation. The function

$$S_u(\nu) = 2 \int_0^{\infty} (\cos \nu\mu)\, \mathscr{L}_u(\mu)\, d\mu$$

has the form of a spectrum; for example, it is nonnegative.

Let s be a nonnegative random variable with a probability density function defined for $\alpha \geqslant 0$

$$p_s(\alpha) \sim \alpha^\delta \qquad \text{small} \quad \alpha \tag{8}$$

where

$$\delta > 0 \tag{9}$$

Now, let

$$w(t) = u(st)$$

thus giving a special form to the random process (6). Namely, let $\{u_i(\theta)\}$ be a sequence of mutually independent random processes, all having the distribution of $u(\theta)$, and let $\{s_i\}$ be a sequence of mutually independent random variables, all having the distribution of s. Assume also that the s_i are independent of the $u_i(\theta)$. Thus

$$n(t) = \sum_{i=-\infty}^{\infty} u_i(s_i(t - t_i)) \tag{10}$$

becomes the process under consideration. It is shown in Appendix B that the random process (10), under the conditions (7)–(9), is well-defined and stationary with mean

$$\bar{n} = \lambda \int_0^\infty (1/\alpha)\, p_s(\alpha)\, d\alpha \int_{-\infty}^{\infty} \bar{u}(\theta)\, d\theta$$

covariance

$$\mathscr{L}_n(\tau) = \lambda \int_0^\infty (1/\alpha)\, p_s(\alpha)\, \mathscr{L}_n(\alpha\tau)\, d\alpha$$

and spectrum

$$S_n(\omega) = \lambda \int_0^\infty (1/\alpha^2)\, p_s(\alpha)\, S_u(\omega/\alpha)\, d\alpha \tag{11}$$

3. 1/f NOISE

3.1. The Limit Spectrum

Consider a probability density function

$$p_0(\alpha) \sim 1 \qquad \text{small} \quad \alpha$$

This does not satisfy (8) and (9), so, replacing p_s in (11) with p_0 allows one to calculate a function of ω, but that function (called the *limit spectrum*) is not necessarily the spectrum of a random process. The limit spectrum is

$$S_0(\omega) = (\lambda/\omega) \int_0^\infty p_0(\alpha)(\omega/\alpha^2)\, S_u(\omega/\alpha)\, d\alpha \tag{12}$$

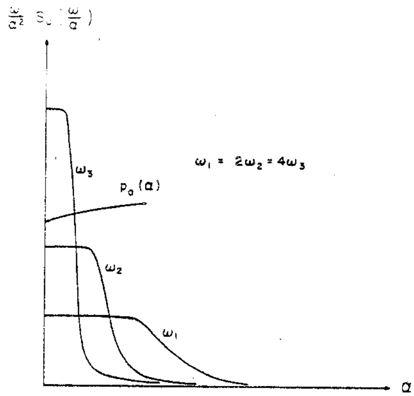

Fig. 1. Plot of $(\omega/\alpha^2)S_u(\omega/\alpha)$ versus α for various ω.

Observe that for $\omega > 0$,

$$\int_0^\infty (\omega/\alpha^2)\, S_u(\omega/\alpha)\, d\alpha = \int_0^\infty S_u(\nu)\, d\nu = \pi \mathscr{L}_u(0)$$

which is a constant, independent of ω. Now, by inspection of Fig. 1, it is clear that as ω is decreased, the function $(\omega/\alpha^2)\, S_u(\omega/\alpha)$, regarded as a function of α, becomes progressively a sharper spike concentrated near $\alpha = 0$, all spikes having constant integral. Hence

$$\int_0^\infty p_0(\alpha)(\omega/\alpha^2)\, S_u(\omega/\alpha)\, d\alpha \approx \pi p_0(0)\, \mathscr{L}_u(0) \qquad \text{small} \quad \omega > 0$$

and

$$S_0(\omega) \approx \pi\lambda p_0(0)\, \mathscr{L}_u(0)/\omega \qquad \text{small} \quad \omega > 0$$

Thus the limit spectrum is strictly $1/f$ for small f.

3.2. Approximation of the Limit Spectrum

Now, reconsider the spectrum of $n(t)$ and make the additional assumption that for some $A > 0$,

$$S_u(\nu) \leqslant A/\nu^2 \tag{13}$$

for all ν. Take any $\omega_0 > 0$ and let $\omega \geqslant \omega_0$. From (11) and (12),

$$\begin{aligned} |\, S_n(\omega) - S_0(\omega)| &= (\lambda/\omega^2) \int_0^\infty |\, p_s(\alpha) - p_0(\alpha)|\, (\omega^2/\alpha^2)\, S_u(\omega/\alpha)\, d\alpha \\ &\leqslant (A\lambda/\omega_0{}^2) \int_0^\infty |\, p_s(\alpha) - p_0(\alpha)|\, d\alpha \end{aligned} \tag{14}$$

from (13), and hence the spectrum of $n(t)$ can be made arbitrarily close to the limit spectrum, uniformly for all $\omega \geqslant \omega_0 > 0$, by taking a probability density $p_s(\alpha)$ very close to $p_0(\alpha)$ in the L_1 sense. This, of course, permits that $S_n(\omega)$ have low-frequency turnover.

3.3. Strict $1/f^{1-\delta}$ Noise

Now, drop the assumption (13) and add the assumption

$$S_u(\nu) \sim 1/\nu^q \qquad \text{small} \quad \nu \tag{15}$$

for some $q < 1 - \delta$. Then from (11), taking $\omega > 0$,

$$\begin{aligned} S_n(\omega) &= (\lambda/\omega^{1-\delta}) \int_0^\infty p_s(\alpha)(\omega^{1-\delta}/\alpha^2)\, S_u(\omega/\alpha)\, d\alpha \\ &= (\lambda/\omega^{1-\delta}) \int_0^\infty h(\alpha)(\omega^{1-\delta}/\alpha^{2-\delta})\, S_u(\omega/\alpha)\, d\alpha \end{aligned}$$

where

$$h(\alpha) = (1/\alpha^\delta)\, p_s(\alpha)$$

so

$$h(\alpha) \sim 1 \qquad \text{small} \quad \alpha$$

Observe that

$$\int_0^\infty (\omega^{1-\delta}/\alpha^{2-\delta})\, S_u(\omega/\alpha)\, d\alpha = \int_0^\infty (1/\nu^\delta)\, S_u(\nu)\, d\nu$$

independently of ω; (15) implies the existence of this integral. By the same sort of argument used in the analysis of $S_0(\omega)$,

$$S_n(\omega) \stackrel{\cdot}{\approx} \pi\lambda h(0)\, I_\delta/\omega^{1-\delta} \qquad \text{small} \quad \omega > 0$$

where

$$\pi I_\delta = \int_0^\infty (1/\nu^\delta)\, S_u(\nu)\, d\nu$$

This is a strict $1/f^{1-\delta}$ spectrum for small f.

4. EXAMPLES

It remains to give physical examples in which the conditions hold. The special case where $u(\theta)$ is a deterministic function,

$$u(\theta) = \begin{cases} 1, & 0 \leqslant \theta \leqslant 1 \\ 0, & \text{elsewhere} \end{cases}$$

is employed. Hence

$$\mathscr{L}_u(\mu) = \begin{cases} 1 - |\mu|, & |\mu| \leqslant 1 \\ 0, & \text{elsewhere} \end{cases}$$

which is a function of bounded variation. Also

$$S_u(\nu) = [(\sin \tfrac{1}{2}\nu)/\tfrac{1}{2}\nu]^2 \tag{16}$$

which satisfies (13).

Consider a rectangular chamber C (Fig. 2) which is adjacent to an infinite source S containing a gas whose molecules have a Gaussian distribution of vector velocity:

$$p_{v_x v_y v_z}(\alpha, \beta, \gamma) = [1/(2\pi)^{3/2}\, \sigma^3] \exp[-(\alpha^2 + \beta^2 + \gamma^2)/2\sigma^2]$$

Molecules enter C through the small hole H and are eventually absorbed by either the wall in the xz plane a distance L from the hole, or one of the two walls in the xy plane a distance W from H. The walls in the yz plane, parallel to the page, are assumed to be reflecting. It is assumed that H is small enough so that there are no collisions of molecules in C; each molecule travels linearly at constant velocity toward one of the absorbing walls.

The random process studied is

$$\begin{aligned} n(t) &= \text{number of molecules in } C \text{ at time } t \\ &= \sum_{i=-\infty}^{\infty} u(s_i(t - t_i)) \end{aligned}$$

where the $\{t_i\}$ are Poisson arrivals, frequency λ, and the $s_i = 1/T_i$, with T_i the time required for a molecule which entered C at time t_i to be absorbed by one of the walls.

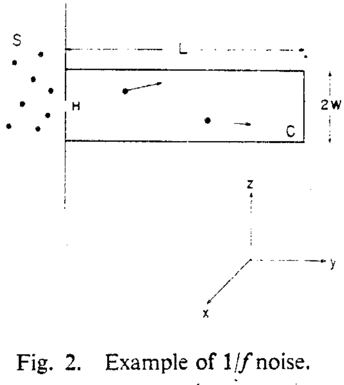

Fig. 2. Example of 1/f noise.

The probability densities of the components of the velocity of a molecule passing through H are

$$p_{v_x}(\alpha) = p_{v_z}(\alpha) = [1/(2\pi)^{1/2}\,\sigma]\,\exp(-\alpha^2/2\sigma^2)$$

$$p_{v_y}(\alpha) = (\alpha/\sigma^2)\exp(-\alpha^2/2\sigma^2), \qquad \alpha \geqslant 0$$

The latter form is on account of the conditioning imposed on v_y by the assumption that the molecule passes through H.

The x component of velocity is irrelevant to T, because of the reflecting nature of the walls in the yz plane. Now,

$$T = \min[W/|v_z|, L/v_y]$$

so

$$s = \max[|v_z|/W, v_y/L]$$

and hence

$$\begin{aligned}
\text{Prob}[s \leqslant \alpha] &= \text{Prob}[|v_z| \leqslant \alpha W]\,\text{Prob}[v_y \leqslant \alpha L] \\
&= [2/(2\pi)^{1/2}\,\sigma]\int_0^{\alpha W} \exp(-\beta^2/2\sigma^2)\,d\beta \\
&\quad \times (1/\sigma^2)\int_0^{\alpha L} \gamma \exp(-\gamma^2/2\sigma^2)\,d\gamma \qquad (17)\\
&= [1 - 2Q(\alpha W/\sigma)][1 - \exp(-\alpha^2 L^2/2\sigma^2)] \\
&\sim \alpha^3, \qquad \text{small} \quad \alpha
\end{aligned}$$

where

$$Q(\mu) = [1/(2\pi)^{1/2}]\int_\mu^\infty \exp(-\nu^2/2)\,d\nu$$

Differentiating to get the probability density,

$$\begin{aligned}
p_s(\alpha) = {} & [2W/(2\pi)^{1/2}\,\sigma]\exp(-\alpha^2 W^2/2\sigma^2) \\
& + [1 - 2Q(\alpha W/\sigma)](\alpha L^2/\sigma^2)\exp(-\alpha^2 L^2/2\sigma^2) \\
& - [2W/(2\pi)^{1/2}\,\sigma]\exp(-\alpha^2 W^2/2\sigma^2)\exp(-\alpha^2 L^2/2\sigma^2)
\end{aligned}$$

for $\alpha \geqslant 0$. Now, take

$$p_0(\alpha) = [2W/(2\pi)^{1/2}\,\sigma]\exp(-\alpha^2 W^2/2\sigma^2), \qquad \alpha \geqslant 0$$

as determining a limit spectrum

$$\begin{aligned}
S_0(\omega) &= (\lambda/\omega)\int_0^\infty [2W/(2\pi)^{1/2}\,\sigma][\exp(-\alpha^2 W^2/2\sigma^2)](\omega/\alpha^2)\,S_u(\omega/\alpha)\,d\alpha \\
&\approx (2\pi)^{1/2}\,(\lambda W/\sigma\omega) \qquad \text{small} \quad \omega
\end{aligned}$$

As an aside, it is remarked that λ/σ is independent of the temperature of the gas in S; $1/f$ noise in semiconductors[2] and in thin gold films[7] also appears to be temperature-independent.

To return to the analysis, since

$$Q(\alpha) \geqslant \tfrac{1}{2} - [1/(2\pi)^{1/2}]\,\alpha, \qquad \alpha \geqslant 0$$

then

$$|p_s(\alpha) - p_0(\alpha)| \leqslant [2WL^2\alpha^2/(2\pi)^{1/2}\,\sigma^3]\exp(-\alpha^2L^2/2\sigma^2) + [2W/(2\pi)^{1/2}\,\sigma]\exp[-\alpha^2(W^2 + L^2)/2\sigma^2]$$

so

$$\int_0^\infty |p_s(\alpha) - p_0(\alpha)|\,d\alpha \leqslant (W/L) + [W/(W^2 + L^2)^{1/2}]$$

Hence the conclusion allowed by (14) is that for any $\omega_0 > 0$, any $\epsilon > 0$, one can take W/L small enough so that $|S_n(\omega) - S_0(\omega)| \leqslant \epsilon$ for all $\omega \geqslant \omega_0$.

To the extent that $L \gg W$, the spectrum of $n(t)$ resembles a $1/f$ spectrum at low frequencies. However, there is eventual low-frequency turnover. Since, from (17),

$$p_s(\alpha) \sim \alpha^2 \qquad \text{small} \quad \alpha$$

and observing from (16) that for small ω, the function $S_u(\omega/\alpha)$ is approximately 1 for most of the range of α, $0 \leqslant \alpha < \infty$, one determines from (11) that

$$S_n(\omega) \approx \lambda \int_0^\infty (1/\alpha^2)\,p_s(\alpha)\,d\alpha \qquad \text{small} \quad \omega$$

Another situation develops in our example if the wall in the xz plane, a distance L from H, is made reflecting. Then, it would appear that

$$p_s(\alpha) = p_0(\alpha) = [2W/(2\pi)^{1/2}\,\sigma]\exp(-\alpha^2W^2/2\sigma^2) \tag{18}$$

but this is not the case, since this p_s implies an unstable $n(t)$. The number of molecules in C grows without bound and collisions eventually occur.

It is possible, although not certain, that H may be made small enough so that the collisions have the effect of modifying p_s only slightly from the form of (18), so that p_s resembles (18) but

$$p_s(\alpha) \sim \alpha^\delta \qquad \text{small} \quad \alpha$$

where $0 < \delta \ll 1$. This, as the analysis has shown, would result in a strict $1/f^{1-\delta}$ spectrum, and δ could be so small that it would be indistinguishable from $1/f$.

The basic idea that has been employed in these examples is that of forcing the time T during which a molecule remains in a given region to be essentially inversely proportional to a velocity component which has not been conditioned. Making $L \gg W$ in the first example means that the absorption by the walls in the xy plane dominates the situation, rendering the conditioned v_y of slight significance. It might be possible to accomplish the latter by introducing a field in the y direction instead of making $L \gg W$.

In fact, it is obvious that the example can be elaborated and complicated considerably, both in regard to geometries and specific physical mechanisms.

Other types of examples which fit the required conditions are no doubt possible. It is likely that the function

$$u(\theta) = \begin{cases} 1, & 0 \leqslant \theta \leqslant 1 \\ 0, & \text{elsewhere} \end{cases} \tag{19}$$

will be prominent in such examples. Also, the probability densities

$$p_{01}(\alpha) = [2/(2\pi)^{1/2}\,\sigma]\exp(-\alpha^2/2\sigma^2), \qquad \alpha \geqslant 0$$

$$p_{02}(\alpha) = (1/\sigma)\exp(-\alpha/\sigma), \qquad \alpha \geqslant 0$$

$$p_{03}(\alpha) = \begin{cases} 1/\sigma, & 0 \leqslant \alpha \leqslant \sigma \\ 0 & \text{elsewhere} \end{cases}$$

will be likewise prominent in determining limit spectra.

In these cases, it is found that $\sigma\omega S_0(\omega)$ is a function of $\mu = \omega/\sigma$. Define $H(\mu) = \sigma\omega S_0(\omega)$. Figure 3 shows $H(\mu)/H(0^+)$ versus μ assuming (19) and the three preceding probability densities.

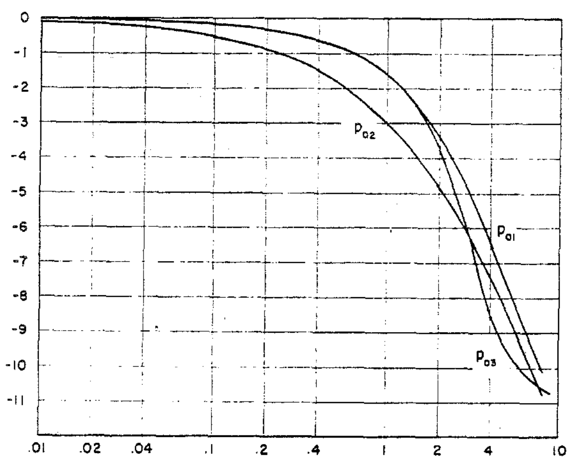

Fig. 3. Plot of $10\log_{10}[H(\mu)/H(0^+)]$ versus μ (log scale) for cases p_{01}, p_{02}, p_{03}.

5. DISCUSSION AND RELATION TO OTHER THEORIES

The noise process studied here is made up of a superposition of pulses of the form $u(s(t - t_i))$, where, to express the result roughly, the condition

$$p_s(\alpha) \sim 1 \qquad \text{small} \quad \alpha \tag{20}$$

implies a $1/f$ spectrum. If one defines $T = 1/s$, then the equivalent pulse is $u((t - t_i)/T)$ and the condition (20) is equivalent to

$$p_T(\alpha) \sim 1/\alpha^2 \qquad \text{large} \quad \alpha \tag{21}$$

Thus a tail of the form (21) implies a $1/f$ spectrum.

As suggested in Section 1, T, rather than s, could have been carried along in the analysis and completely equivalent mathematical results would have been obtained. The motivation for choosing s should be clear from the material in Section 4; it seems that the problem of finding specific physical situations meeting the conditions of the theory is facilitated thereby. The reason for this is simply that if one considers the known probability laws associated with the variables on which attention is conventionally focused, the probability densities of the form (20) far outnumber those of the form (21). This point is, of course, empty in a strictly mathematical sense. One could have carried T, rather than s, along in the analysis and one could have, in Section 4, observed that the probability densities of the reciprocals of the velocity components were of the form

$$[1/(2\pi)^{1/2} \sigma\alpha^2] \exp(-1/2\sigma^2\alpha^2)$$

and then reached the same conclusions. This would clearly have been an unnatural and awkward procedure.

The significance of the example presented in Section 4 is twofold. First, it seems to be the first such construction of a simple physical situation in which the $1/f$ spectrum is manifested. That is to say, no arbitrary assumptions were made to get the desired result; neither a strange waveform nor an unfounded probability density was required. Second, the example shows that the theory is not vacuous. There is little doubt that some, if not all, $1/f$ phenomena are explicable in terms of the present theory.

However, there is no intention to suggest that the specific example presented in Section 4 is the key to explaining the $1/f$ spectrum in any of the physical contexts mentioned in the introduction; for those problems, all that can be said is that a mathematical theory of significant generality and simplicity is offered. It is hoped that persons with expertise in those contexts will attempt to apply the theory in more specific ways.

In order to relate the present theory to others, the result equivalent to (11) is now derived in a somewhat looser manner. If a random process $X(t)$ is made up of pulses of the form

$$u((t - t_i)/T) \tag{22}$$

where the $\{t_i\}$ are Poisson arrivals and T is fixed (the same for all i), then its covariance function is, by Campbell's theorem, proportional to

$$\int_{-\infty}^{\infty} u(t/T)\, u((t + \tau)/T)\, dt = T\mathscr{L}_u(\tau/T)$$

If

$$u(\theta) = \begin{cases} e^{-\theta}, & \theta \geqslant 0 \\ 0 & \theta < 0 \end{cases} \tag{23}$$

then the covariance function is proportional to

$$\tfrac{1}{2}T \exp(-|\tau|/T) \tag{24}$$

The spectrum of this process is proportional to

$$2T \int_0^{\infty} (\cos \omega\tau)\, \mathscr{L}_u(\tau/T)\, d\tau = T^2 S_u(\omega T)$$

and under the condition (23), then, the spectrum is proportional to

$$T^2/(1 + \omega^2 T^2) \tag{25}$$

If it is assumed that the random process $n(t)$ is a superposition of random processes of this sort, but having different time constants T with probability density $p_T(\alpha)$, then the spectrum is proportional to

$$\int_0^{\infty} p_T(\alpha)\, \alpha^2 S_u(\alpha\omega)\, d\alpha \tag{26}$$

The result (26) is completely equivalent to (11) when the relation $s = 1/T$ is made. If (23) holds, then the spectrum is proportional to

$$\int_0^{\infty} p_T(\alpha)[\alpha^2/(1 + \omega^2\alpha^2)]\, d\alpha \tag{27}$$

and it is seen that if (21) holds, then this gives a $1/f$ spectrum for small f.

The reasoning here is almost identical to that of van der Ziel,[9] except that in place of (27), van der Ziel has (except for differences in notation)

$$4\overline{X(t)^2} \int_0^{\infty} p_T(\alpha)[\alpha/(1 + \omega^2\alpha^2)]\, d\alpha \tag{28}$$

[Eq. (11) in Ref. 9], which obviously calls for a $p_T(\alpha)$ of the form $1/\alpha$ (over some finite interval) if an approximate $1/f$ spectrum is to be obtained.

The discrepancy is due to the manner in which $\overline{X(t)^2}$ is treated. If (23), holds then from (24).

$$\overline{X(t)^2} = T/2$$

so $\overline{X(t)^2}$ should be inside the integral of (28), thereby giving our result (27). Expression (28) is correct only if constant "energy" pulses of the form

$$(1/\sqrt{T})\, u((t - t_i)/T)$$

are assumed in the model, in place of (22). This assumption was not stated in Ref. 9, but it is necessary if a $p_T(\alpha)$ of the form $1/\alpha$ is to imply a $1/f$ spectrum.

The same sort of situation appears to hold in Ref. 12 and, thus, also in Ref. 13, which refers to Ref. 12 on this point. In Ref. 13, the $1/\alpha$ form is again required for $p_T(\alpha)$, for under the condition that

$$T = T_0 e^{Ax}$$

it is assumed that

$$p_x(\alpha) = \text{const}, \qquad 0 \leqslant x \leqslant x_1$$

which is equivalent.

This matter is not pursued further here because there is no intention of arguing the correctness or incorrectness of the van der Ziel theory, and in any case van der Ziel's objectives have differed from those pursued here. It is only desired to show the relation of the present theory to a portion of van der Ziel's and to indicate the exact point of divergence.

It is worth remarking that no probability density may have the form $1/\alpha$ for small α or as a tail, and thus, no concept of a limit spectrum, paralleling the present concept, is possible in a theory which requires that $p_T(\alpha)$ have the form $1/\alpha$ over some finite range.

6. CONCLUSION

The result of this paper is a mathematical theory of $1/f$ noise which is related to that of van der Ziel, but which differs on one basic point, is somewhat more general, and appears to offer purely mathematical advantages. The theory is such that it is possible to construct simple physical situations exhibiting the $1/f$ spectrum without making unfounded assumptions or arbitrary postulates.

A special result is the introduction of the concept of a limit spectrum for $1/f$ noise. This is paralleled in the case of white noise, where the associated spectrum also represents a limit which may not be physically attained.

APPENDIX A

First recall Campbell's theorem. Let $v(t)$ be some function and let a random process

$$m(t) = \sum_{i=-\infty}^{\infty} v(t - t_i)$$

be defined, where the $\{t_i\}$ are Poisson arrivals with average frequency λ. Campbell's theorem then asserts that

$$\overline{m} = E\{m(t)\} = \lambda \int_{-\infty}^{\infty} v(t)\, dt$$

$$E\{m(t)\, m(t + \tau)\} = \lambda \int_{-\infty}^{\infty} v(t)\, v(t + \tau)\, dt + \overline{m}^2$$

Now, under the assumption that

$$\int_{-\infty}^{\infty} \overline{w}_a(t)\, dt < \infty \tag{A.1}$$

the process $n(t)$ is well-defined and stationary, for the random variable $n(t)$ is well-defined if $|n(t)| < \infty$ with probability one. This is implied by the condition $E\{|n(t)|\} < \infty$, but

$$|n(t)| \leqslant r(t) = \sum_{i=-\infty}^{\infty} |w_i(t - t_i)|$$

and

$$E\{r(t)\} = E\left\{\sum_{i=-\infty}^{\infty} E\{w_a(t - t_i)|\, t_i\}\right\} = E\left\{\sum_{i=-\infty}^{\infty} \overline{w}_a(t - t_i)\right\}$$
$$= \lambda \int_{-\infty}^{\infty} \overline{w}_a(t)\, dt$$

by Campbell's theorem. Hence (A.1) implies that the process (6) is well-defined. Now, defining a random process

$$n'(t) = \sum_{i=-\infty}^{\infty} \overline{w}(t - t_i)$$

one has

$$E\{n(t)\} = E\{n'(t)\} = \lambda \int_{-\infty}^{\infty} \overline{w}(t)\, dt = \overline{n}$$

and

$$E\{n'(t)\,n'(t+\tau)\} = E\left\{\sum_{i,j} \bar{w}(t-t_i)\,\bar{w}(t+\tau-t_j)\right\}$$
$$= \bar{n}^2 + \lambda \int_{-\infty}^{\infty} \bar{w}(t)\,\bar{w}(t+\tau)\,dt$$

but

$$E\left\{\sum_{i=-\infty}^{\infty} \bar{w}(t-t_i)\,\bar{w}(t+\tau-t_i)\right\} = \lambda \int_{-\infty}^{\infty} \bar{w}(t)\,\bar{w}(t+\tau)\,dt$$

so

$$E\left\{\sum_{i\neq j} \bar{w}(t-t_i)\,\bar{w}(t+\tau-t_j)\right\} = \bar{n}^2$$

Now,

$$E\{n(t)\,n(t+\tau)\} = E\left\{\sum_{i=-\infty}^{\infty} w_i(t-t_i)\,w_i(t+\tau-t_i)\right\}$$
$$+ E\left\{\sum_{i\neq j} \bar{w}(t-t_i)\,\bar{w}(t+\tau-t_j)\right\}$$

by the mutual independence of the $\{w_i\}$, so

$$E\{n(t)\,n(t+\tau)\} = E\left\{\sum_{i=-\infty}^{\infty} E\{w(t-t_i)\,w(t+\tau-t_i)|\,t_i\}\right\} + \bar{n}^2$$
$$= \lambda \int_{-\infty}^{\infty} E\{w(t)\,w(t+\tau)\}\,dt + \bar{n}^2$$

Hence, under the assumption (A.1), the random process $n(t)$ of (6) is well-defined, stationary, with mean

$$\bar{n} = \lambda \int_{-\infty}^{\infty} \bar{w}(t)\,dt$$

and covariance

$$\mathscr{L}_n(\tau) = \lambda \mathscr{L}_w(\tau)$$

where $\bar{w}$ is defined in (4) and $\mathscr{L}_w$ in (5).

APPENDIX B

Since with $w(t) = u(st)$,

$$\int_{-\infty}^{\infty} \bar{w}_a(t)\,dt = \int_{-\infty}^{\infty} \int_{0}^{\infty} E\{u_a(st)|\,s=\alpha\}\,p_s(\alpha)\,d\alpha\,dt$$
$$= \int_{-\infty}^{\infty} \int_{0}^{\infty} \bar{u}_a(\alpha t)\,p_s(\alpha)\,d\alpha\,dt$$
$$= \int_{0}^{\infty} (1/\alpha)\,p_s(\alpha)\,d\alpha \int_{-\infty}^{\infty} \bar{u}_a(\theta)\,d\theta < \infty$$

by (7)–(9) and the mutual independence of s and u, then by the results of Appendix A, the random process $n(t)$ of (10) is well-defined and stationary and

$$\bar{n} = \lambda \int_{-\infty}^{\infty} \bar{w}(t)\,dt = \lambda \int_{-\infty}^{\infty} \int_{0}^{\infty} E\{u(st) \mid s = \alpha\}\, p_s(\alpha)\, d\alpha\, dt$$

$$= \lambda \int_{0}^{\infty} \int_{-\infty}^{\infty} \bar{u}(\alpha t)\, p_s(\alpha)\, dt\, d\alpha = \lambda \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha)\, d\alpha \int_{-\infty}^{\infty} \bar{u}(\theta)\, d\theta$$

and

$$\mathscr{L}_n(\tau) = \lambda \int_{-\infty}^{\infty} E\{w(t)\, w(t+\tau)\}\, dt$$

$$= \lambda \int_{-\infty}^{\infty} \int_{0}^{\infty} E\{u(st)\, u(st + s\tau) \mid s = \alpha\}\, p_s(\alpha)\, d\alpha\, dt$$

$$= \lambda \int_{0}^{\infty} p_s(\alpha) \int_{-\infty}^{\infty} E\{u(\alpha t)\, u(\alpha t + \alpha\tau)\}\, dt\, d\alpha$$

$$= \lambda \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha)\, \mathscr{L}_u(\alpha\tau)\, d\alpha$$

Since $\mathscr{L}_u$ is bounded, this integral exists, by (8) and (9). In calculating the spectrum of $n(t)$, one must exercise care in reversing orders of integration:

$$S_n(\omega) = \lim_{T \to \infty} 2\lambda \int_{0}^{T} (\cos \omega\tau) \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha)\, \mathscr{L}_u(\alpha\tau)\, d\alpha\, d\tau$$

$$= \lim_{T \to \infty} 2\lambda \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha) \int_{0}^{T} (\cos \omega\tau)\, \mathscr{L}_u(\alpha\tau)\, d\tau\, d\alpha$$

Now,

$$\int_{0}^{T} (\cos \omega\tau)\, \mathscr{L}_u(\alpha\tau)\, d\tau = (1/\alpha) \int_{0}^{\alpha T} [\cos(\omega\mu/\alpha)]\, \mathscr{L}_u(\mu)\, d\mu$$

so for all $\alpha > 0$, $T > 0$, by the lemma in Appendix C,

$$\left| \int_{0}^{T} (\cos \omega\tau)\, \mathscr{L}_u(\alpha\tau)\, d\tau \right| \leqslant K/\omega$$

for some $K > 0$. Hence by the dominated convergence theorem and (8) and (9)

$$S_n(\omega) = 2\lambda \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha) \int_{0}^{\infty} (\cos \omega\tau)\, \mathscr{L}_u(\alpha\tau)\, d\tau\, d\alpha$$

$$= \lambda \int_{0}^{\infty} (1/\alpha)\, p_s(\alpha)(1/\alpha)\, S_u(\omega/\alpha)\, d\alpha$$

APPENDIX C

A function $g(\mu)$ defined on $[0, \infty)$ is of bounded variation if and only if there exists a constant M_0, $0 < M_0 < \infty$, such that for any N and any $\{\mu_i\}$ such that

$$0 \leqslant \mu_0 < \mu_1 < \mu_2 \cdots < \mu_N < \infty$$

it is the case that

$$\sum_{i=1}^{N} | g(\mu_i) - g(\mu_{i-1})| \leqslant M_0$$

Such a function is necessarily bounded;

$$| g(\mu)| \leqslant M_1 < \infty$$

on $[0, \infty)$.

Lemma. Let $g(\mu)$ be a continuous function of bounded variation on $[0, \infty)$ with bounds M_0, M_1. Then, for any ν and $T > 0$,

$$\left| \int_0^T (\cos \nu\mu)\, g(\mu)\, d\mu \right| \leqslant (M_0 + M_1)/\nu$$

Proof. By the mean value theorem for integrals and the continuity of g,

$$\int_0^T (\cos \nu\mu)\, g(\mu)\, d\mu$$

$$= g(\beta_0) \int_0^{\pi/2\nu} \cos(\nu\mu)\, d\mu + g(\beta_1) \int_{\pi/2\nu}^{3\pi/2\nu} \cos(\nu\mu)\, d\mu$$

$$+ g(\beta_2) \int_{3\pi/2\nu}^{5\pi/2\nu} \cos(\nu\mu)\, d\mu + \cdots + g(\beta_n) \int_{(2n-1)\pi/2\nu}^{(2n+1)\pi/2\nu} \cos(\nu\mu)\, d\mu$$

$$+ g(\beta_{n+1}) \int_{(2n+1)\pi/2\nu}^{T} \cos(\nu\mu)\, d\mu$$

where n is such that

$$\pi/\nu \geqslant T - [(2n + 1)\,\pi/2\nu] > 0$$

and

$$0 < \beta_0 < \pi/2\nu, \qquad (2i - 1)\,\pi/2\nu < \beta_i < (2i + 1)\,\pi/2\nu, \qquad i = 1, 2, \ldots, n$$

$$(2n + 1)\,\pi/2\nu < \beta_{n+1} < T$$

Hence

$$\int_0^T (\cos \nu\mu)\, g(\mu)\, d\mu = (1/\nu)[g(\beta_0) - 2g(\beta_1) + 2g(\beta_2) \cdots \pm 2g(\beta_n) \mp g(\beta_{n+1}) \mp cg(\beta_{n+1})]$$

where $|c| \leqslant 1$. But then

$$\left| \int_0^T (\cos \nu\mu)\, g(\mu)\, d\mu \right| \leqslant (1/\nu) \sum_{i=1}^{n+1} |g(\beta_i) - g(\beta_{i-1})| + (1/\nu)|g(\beta_{n+1})| \leqslant (M_0 + M_1)/\nu$$

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